

RESEARCH PAPERS

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APPROXIMATE NUMERICAL MODEL FOR ADHESIVE SCARF JOINTS IN WOODEN BEAMS

The purpose of this paper is to formulate an approximate numerical model for adhesive scarf joints in wooden beams under the assumption that the joint is deformable. This approximate model is founded on observations resulting from the general solution of the plane linear theory of elasticity. It is assumed that wood is orthotropic. The joint can be subjected to a complex load combination including an axial force, a shear force and a bending moment. Within this framework, analytical relations for stresses in the adhesive, as well as stresses and displacements of the adherends, were derived in a form useful for engineering practice. A numerical example proves the high accuracy of the approximate model when compared to the general model based on the plane linear theory of elasticity.

Keywords: adhesive scarf joint, analytical model, stress analysis, orthotropy

Introduction

The model of a scarf joint presented in the paper by Rapp [2014] was expressed by a set of four partial differential equations of the second order. The boundary conditions included the presence of the sharp edges of the adherends. Examples of complete solutions within the framework of the theory of elasticity were given and analysed, with a focus on how the adhesive joint transfers axial forces, bending moments and shear forces. It was shown that the adhesive scarf joint does not feature stress concentrations in the adhesive and that there exists an approximate equivalence of displacement and stress states between the element made of two adherends connected by a deformable scarf joint and the continuous element considered as one with a non-deformable scarf joint.

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The continuous element represents an approximation of the deformable scarf joint only when adhesive deformability is small. However, this is not always the case. For instance, in repair works carried out on-site, wooden beam elements are usually reconstructed by cutting off a damaged fragment and replacing it with a new element matching the old one (the exchange of the damaged end of a floor beam, a purlin or a rafter). In construction site conditions, cutting is done using a chain saw. This yields surfaces which are unsmooth and are not flat. Gaps between the old and new elements have a varying thickness of 2–3 mm and are filled with adhesive, e.g. a resin composite of small viscosity. Thus, a thick adhesive joint is formed with a relatively high deformability.

These types of joints are rarely used in wooden beam structures and as technological solutions, they do not yet possess either a complete theoretical or experimental documentation. In other cases, differing from the ones described in this paper, scarf joints are frequently used under axial tension [Erdogan, Ratawani 1971; Reddy, Sinha 1975] or in beams under bending, as joints in the form of micro-dovetails (finger joints), where the joint is perpendicular to the bending plane [Tomasiuk 1988; Smardzewski 1996].

The purpose of this paper is to formulate and use an approximate numerical model for an adhesive scarf joint with a deformable adhesive. Thus, the problem of solving the set of four partial differential equations given by Rapp [2014] is avoided, and analytical formulae to find stresses in the adhesive, as well as stresses and displacements in the adherends, are obtained in a form useful in engineering practice.

Scarf joint model

A wooden beam of a rectangular cross-section is considered consisting of two adherends of thickness g made of the same material and connected by an adhesive scarf joint. The scarf joint features the adhesive of a plane rectangular shape forming an angle φ_x with the plane of beam bending OXY (fig. 1).

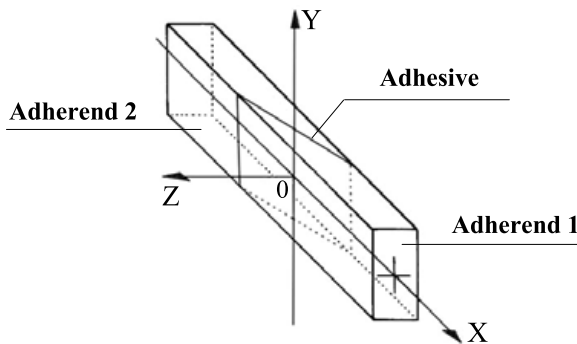


Fig. 1. Adhesive scarf joint in a wooden beam

The set of co-ordinates $0XYZ$ is attached to the beam and $0XY$ is the plane of beam bending. The adhesive plane is parallel to the Y axis. The projection of the adhesive plane on the plane $0XY$ is the rectangle $ABCD$ with the dimensions $2l_x \times 2l_y$ (fig. 2).

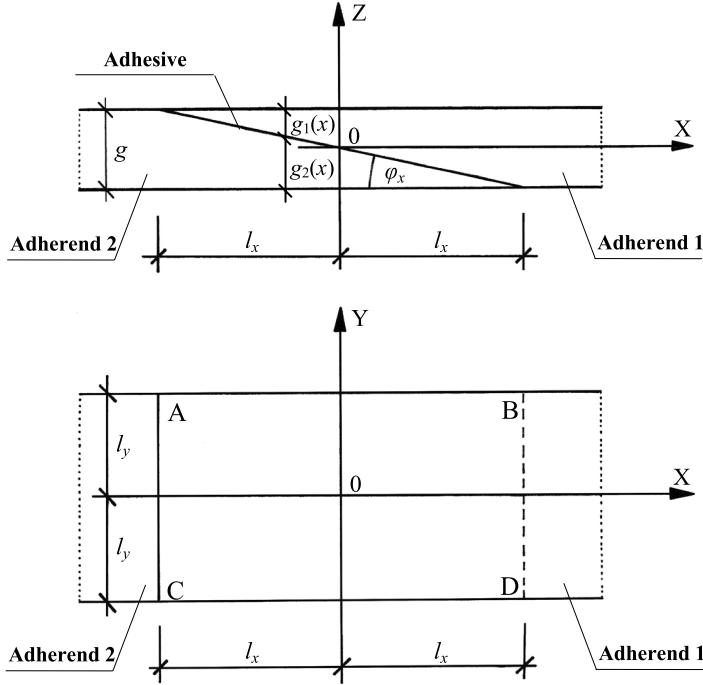


Fig. 2. The adhesive scarf joint model

It is assumed that stresses are constant across the adherend thickness and form a plane stress state parallel to the plane $0XY$. The adherends are considered as plane stress elements parallel to the plane $0XY$. The adherend thickness is measured perpendicularly to the plane $0XY$, and the adhesive thickness t is measured perpendicularly to its plane.

In the joint zone, the thickness values $g_1(x)$, $g_2(x)$ of adherends 1 and 2 vary linearly from 0 to g along the X axis and are constant along the Y axis:

$$g_1(x) = \frac{g}{2l_x}x + \frac{g}{2}, \quad g_2(x) = -\frac{g}{2l_x}x + \frac{g}{2} \quad (1)$$

The adhesive is modelled as a linearly elastic isotropic medium described with Young's modulus E_s , the shear modulus G_s and Poisson's ratio ν_s , where $E_s = 2(1 + \nu_s)G_s$. The stress in the adhesive is defined as the action of adherend 1 on the adhesive. The adhesive is subjected to shear stress $\tau_x = \tau_x(x, y)$, $\tau_y = \tau_y(x, y)$

in its plane and normal stress $\sigma_N = \sigma_N(x, y)$ perpendicular to its plane. Stress τ_x is parallel to the plane $0XZ$, and stress τ_y is parallel to the Y axis (fig. 3).

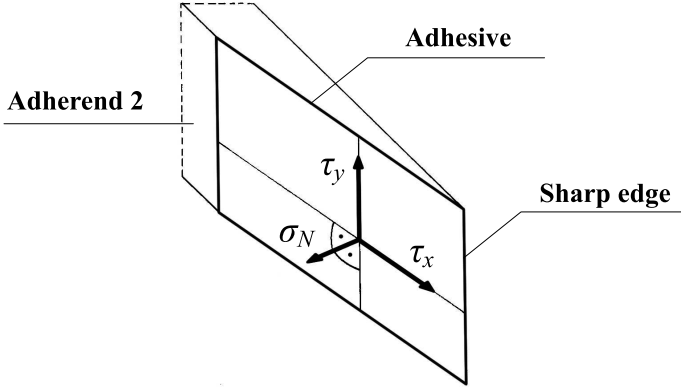


Fig. 3. Stresses in the adhesive

It is assumed that the stress in the adhesive is constant across its thickness. The sign convention for the stresses is presented in fig. 3.

The resultant from the shear stress τ_x and the normal stress σ_N in the adhesive is denoted by p_x . Stress p_x is parallel to the X axis and can be given as:

$$p_x = \tau_x \cos \varphi_x + \sigma_N \sin \varphi_x \quad (2)$$

As a result of the action of shear stresses τ_x and τ_y , a shear deformation of the adhesive occurs which causes relative displacements of the adhesive layers in the direction parallel to its plane. Stress σ_N results in adhesive strain normal to its plane.

Displacements in adherends 1 and 2 of the scarf joint are given as functions $u_1 = u_1(x, y)$, $u_2 = u_2(x, y)$ in the X axis direction and $v_1 = v_1(x, y)$, $v_2 = v_2(x, y)$ in the Y axis direction. Displacements u_1 , u_2 , v_1 , v_2 are positive when coinciding with the X and Y axes orientations. It is assumed that the functions u_1 , u_2 , v_1 , v_2 are C^2 -continuous in the sense of the partial derivatives with respect to the variables x, y .

In the continuation, the functions of displacements u_1 , u_2 , v_1 , and v_2 will be considered unknown and all the quantities related to the adhesive joint will be expressed in terms of these functions.

The scarf joint is considered a section of the beam with the length $2l_x$ loaded by an axial force N , a shear force T and a bending moment M . It is approximately assumed that the joint is loaded by the internal forces N , T , and M existing at the midpoint of the scarf joint. The states of displacements and stress in the adherends of the scarf joint resulting from forces N , T , and M are determined using the superposition rule.

Constitutive equations for the adherends

It is assumed that the adherends are made of the same orthotropic wood with the principal axes of orthotropy coinciding with the X and Y axes. In the plane stress state, the constitutive relations for adherends 1 and 2 are given by:

$$\varepsilon_{kx} = \frac{1}{E_x} \sigma_{kx} - \frac{\nu_{xy}}{E_y} \sigma_{ky} \quad (3)$$

$$\varepsilon_{ky} = -\frac{\nu_{yx}}{E_x} \sigma_{kx} + \frac{1}{E_y} \sigma_{ky} \quad (4)$$

$$\gamma_{kxy} = \frac{1}{G_{xy}} \tau_{kxy} \quad (5)$$

where $k = 1$ for adherend 1 and $k = 2$ for adherend 2.

An orthotropic material in the plane stress state is defined by five material parameters: two longitudinal moduli of elasticity E_x and E_y , one shear modulus G_{xy} and two Poisson's ratios ν_{xy} and ν_{yx} . It is assumed that the matrix of coefficients of the set of equations (3) – (4) is symmetric, i.e.:

$$\frac{\nu_{yx}}{E_x} = \frac{\nu_{xy}}{E_y} \quad (6)$$

holds. Hence, only four out of five material parameters are independent.

Having solved (3) – (5) with respect to stresses, one gets:

$$\sigma_{kx} = \frac{E_x}{1 - \nu_{xy} \nu_{yx}} \varepsilon_{kx} + \frac{\nu_{xy} E_x}{1 - \nu_{xy} \nu_{yx}} \varepsilon_{ky} \quad (7)$$

$$\sigma_{ky} = \frac{\nu_{yx} E_y}{1 - \nu_{xy} \nu_{yx}} \varepsilon_{kx} + \frac{E_y}{1 - \nu_{xy} \nu_{yx}} \varepsilon_{ky} \quad (8)$$

$$\tau_{kxy} = G_{xy} \gamma_{kxy} \quad (9)$$

where $k = 1$ for adherend 1 and $k = 2$ for adherend 2.

Particular values of the material parameters E_x , E_y , G_{xy} , ν_{xy} and ν_{yx} can be found in [Keylwerth 1951; Goodman, Bodig 1970; Neuhaus 1994]. In the following calculations, the data for spruce are assumed: $E_x = 1.2 \cdot 10^6$ N/cm², $E_y = 0.8 \cdot 10^5$ N/cm², $G_{xy} = 0.6 \cdot 10^5$ N/cm², $\nu_{xy} = 0.03$, and $\nu_{yx} = 0.45$.

Constitutive equations for the adhesive

Stresses in the adhesive result from relative displacements between adherends 1 and 2. More general considerations make it possible to write the relations between the stresses in the adhesive and the displacements of the adherends in the following form:

$$\tau_x = \frac{E_s G_s}{t(E_s + G_s \operatorname{tg}^2 \varphi_x) \cos \varphi_x} (u_1 - u_2) \quad (10)$$

$$\tau_y = \frac{G_s}{t} (\nu_1 - \nu_2) \quad (11)$$

$$\sigma_N = \tau_x \operatorname{tg} \varphi_x \quad (12)$$

Scarf joint loaded axially

Let us consider a scarf joint loaded by an axial force N , with tension considered positive. The area of the limiting cross-sections of the adherends $x = \pm l_x$ is $A = 2gl_y$ (fig. 2). Using a semi-inverse method within the framework of the plane linear theory of elasticity [Rapp 2014], one can prove that the functions of displacements of the adherends in the considered scarf joint given in the set of co-ordinates OXY (fig. 2) have the following form:

$$u_1(x, y) = \frac{N}{AE_x} x \quad (13)$$

$$u_2(x, y) = \frac{N}{AE_x} x - \frac{\operatorname{tg}(E_s + G_s \operatorname{tg}^2 \varphi_x) \cos^3 \varphi_x}{2l_x G_s E_s} \cdot \frac{N}{A} \quad (14)$$

$$\nu_1(x, y) = \nu_2(x, y) = -\frac{\nu_{yx} N}{AE_x} y \quad (15)$$

and the stresses in the adherends read:

$$\sigma_{1x}(x, y) = \sigma_{2x}(x, y) = \frac{N}{A} = \text{const} \quad (16)$$

$$\sigma_{1y}(x, y) = \sigma_{2y}(x, y) = 0 \quad (17)$$

$$\tau_{1xy}(x, y) = \tau_{2xy}(x, y) = 0 \quad (18)$$

Knowing the displacement functions (13)–(15), one can determine the stresses in the adhesive τ_x , τ_y and σ_N using (10)–(12) to get:

$$\tau_x(x, y) = \frac{N}{A} \sin \varphi_x \cos \varphi_x = \text{const} \quad (19)$$

$$\tau_y(x, y) = 0 \quad (20)$$

$$\sigma_N(x, y) = \frac{N}{A} \sin^2 \varphi_x = \text{const} \quad (21)$$

From relations (19)–(21) it is found that the axially loaded scarf joint between adherends made of the same material features stress states in the adherends and the adhesive independent of the adhesive thickness and the material properties, which are identical to the ones in a cross-section at angle ϑ_x in a continuous element under a uniaxial stress state.

The adhesive parameters only influence the difference between the displacements of adherends 1 and 2, which is evident in relations (13) and (14).

Scarf joint loaded by a shear force

The shear force T is considered positive when its orientation coincides with the one on the Y axis. According to the solution of the theory of elasticity equations, the shear forces T in the cross-section $x = -l_x$ of adherend 2 and $-T$ in the cross-section $x = l_x$ of adherend 1 cause shear stress τ_{xy} in the adherends and the shear stress τ_y in the adhesive. The stresses have parabolic distribution with an accuracy of 0.25% [Rapp 2014]. Due to this fact, it is assumed in the approximate solution that the shear stresses in the adherends and the adhesive are parabolic and are given by:

$$\tau_{1,xy}(x, y) = \tau_{2,xy}(x, y) = \frac{3T(l_y^2 - y^2)}{4gl_y^3} \quad (22)$$

$$\tau_y(x, y) = \frac{3T(l_y^2 - y^2)}{4l_y^3 \sqrt{4l_x^2 + g^2}} \quad (23)$$

Scarf joint loaded by a bending moment

Let us consider now the scarf joint loaded by the bending moments M in the form of linearly distributed stress at the edge $x = l_x$ of adherend 1 and the edge $x = -l_x$ of adherend 2, according to fig. 4.

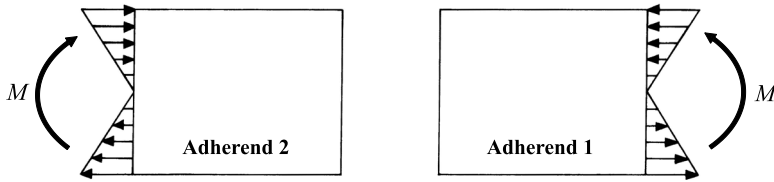


Fig. 4. Loading acting on adherends of a scarf joint subjected to a moment

Figs 5 and 6 present the normal and shear stress distributions in the adherends and the adhesive of the scarf joint resulting from the solution of the plane stress linear theory of elasticity. From now on, this solution will be considered exact and will form the basis for the assumptions to derive an approximate method of solution for the joint loaded by moments. The exact solution concerns the scarf joint loaded by the moments $M = 1 \text{ N}\cdot\text{cm}$ and made of spruce wood with the material parameters $E_x = 1.2 \cdot 10^6 \text{ N/cm}^2$, $E_y = 0.8 \cdot 10^5 \text{ N/cm}^2$, $G_{xy} = 0.6 \cdot 10^5 \text{ N/cm}^2$, $\nu_{xy} = 0.03$, and $\nu_{yx} = 0.45$, and the dimensions $l_x = 22.5 \text{ cm}$, $l_y = 10.25 \text{ cm}$, and $g = 4.5 \text{ cm}$. For the adhesive, the thickness $t = 0.05 \text{ cm}$ and the material parameters $E_s = 1.215 \cdot 10^5 \text{ N/cm}^2$, $G_s = 0.45 \cdot 10^5 \text{ N/cm}^2$, and $\nu_s = 0.35$ were assumed. The stress values at characteristic points of the adhesive and the adherends are given in tables 1 and 2.

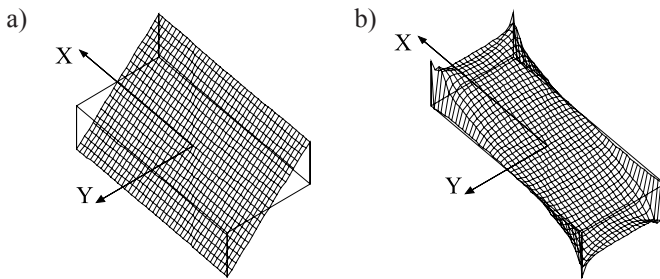


Fig. 5. Stresses in adhesive of a joint loaded by moments: a) stress τ_x , b) stress τ_y (values $\times 1500$)

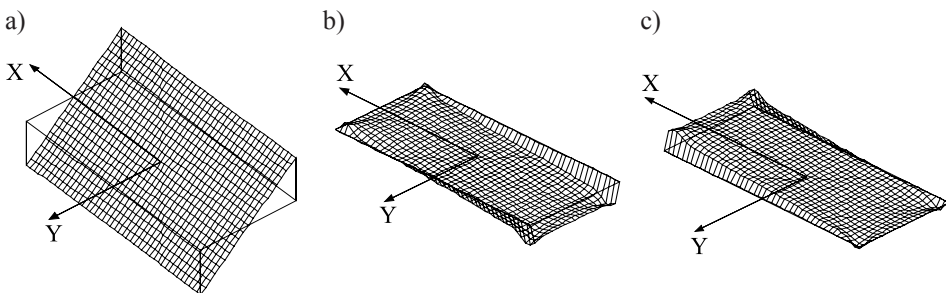


Fig. 6. Stresses in adherends of a joint loaded by moments: a) stresses σ_{1x} , σ_{2x} , b) stress τ_{1xy} (values $\times 500$), c) stress τ_{2xy} (values $\times 500$)

Table 1. Stresses in adhesive of a joint loaded by a moment

$y \setminus x$	N/cm ²	$x = -l_x$	$x = 0$	$x = l_x$
$y = l_y$	τ_x	$-3.132 \cdot 10^{-4}$	$-3.105 \cdot 10^{-4}$	$-3.132 \cdot 10^{-4}$
	τ_y	$-3.240 \cdot 10^{-7}$	0	$3.240 \cdot 10^{-7}$
$y = 0$	τ_x	0	0	0
	τ_y	$-2.031 \cdot 10^{-7}$	0	$2.031 \cdot 10^{-7}$
$y = -l_y$	τ_x	$3.132 \cdot 10^{-4}$	$3.105 \cdot 10^{-4}$	$3.132 \cdot 10^{-4}$
	τ_y	$-3.240 \cdot 10^{-7}$	0	$3.240 \cdot 10^{-7}$

Table 2. Stresses in adherends of a scarf joint loaded by a moment

$y \setminus x$	N/cm ²	Adherend 1			Adherend 2		
		$x = -l_x$	$x = 0$	$x = l_x$	$x = -l_x$	$x = 0$	$x = l_x$
$y = l_y$	σ_x	$-3.166 \cdot 10^{-3}$	$-3.175 \cdot 10^{-3}$	$-3.173 \cdot 10^{-3}$	$-3.173 \cdot 10^{-3}$	$-3.175 \cdot 10^{-3}$	$-3.166 \cdot 10^{-3}$
	τ_{xy}	0	0	0	0	0	0
$y = 0$	σ_x	0	0	0	0	0	0
	τ_{xy}	$-1.954 \cdot 10^{-6}$	$-1.020 \cdot 10^{-6}$	0	0	$1.020 \cdot 10^{-6}$	$1.954 \cdot 10^{-6}$
$y = -l_y$	σ_x	$3.166 \cdot 10^{-3}$	$3.175 \cdot 10^{-3}$	$3.173 \cdot 10^{-3}$	$3.173 \cdot 10^{-3}$	$3.175 \cdot 10^{-3}$	$3.166 \cdot 10^{-3}$
	τ_{xy}	0	0	0	0	0	0

$\min \tau_{1xy} = -2.940 \cdot 10^{-6}$ N/cm² (fig. 6b), $\max \tau_{2xy} = 2.940 \cdot 10^{-6}$ N/cm² (fig. 6c), $\sigma_{1y} = \sigma_{2y} = 0$

From now on, the adhesive stresses p_x and q_y parallel to the plane OXY are considered, where $p_x = \tau_x \cos \varphi_x + \sigma_N \sin \varphi_x$ and $q_y = \tau_y$. From fig. 5, in which the solutions of the theory of elasticity equations are presented, which are considered as exact in this paper, it is assumed in the approximate solution that the stress distributions p_x and q_y are determined by the planes passing through the X and Y axes, respectively, as shown in fig. 7.

Let us denote $p = \max p_x(x, y)$ and $q = \max q_y(x, y)$. The stresses p_x and p_y in the adopted set of co-ordinates are given by:

$$p_x(x, y) = \frac{p}{l_y} y, \quad q_y(x, y) = \frac{q}{l_x} x \quad (24)$$

The stress resultant in the adhesive must be in equilibrium with the moment M . Hence, the following condition yields:

$$p \cdot l_y + q \cdot l_x = \frac{3M}{4l_x l_y} \quad (25)$$

The shear force $Q(x)$ in adherend 1 due to the loading q_y is determined as:

$$Q(x) = \frac{q \cdot l_y}{l_x} (x - l_x)(x + l_x) \quad (26)$$

Taking into account the relation $g(x) = \frac{g}{2l_x}(x + l_x)$ for adherend 1 thickness, one can approximately assume, based on fig. 7b, that the shear stress τ_{1xy} in adherend 1 is:

$$\tau_{1xy}(x, y) = \frac{Q(x)}{2g(x)l_y} = \frac{q(x - l_x)}{g} \quad (27)$$

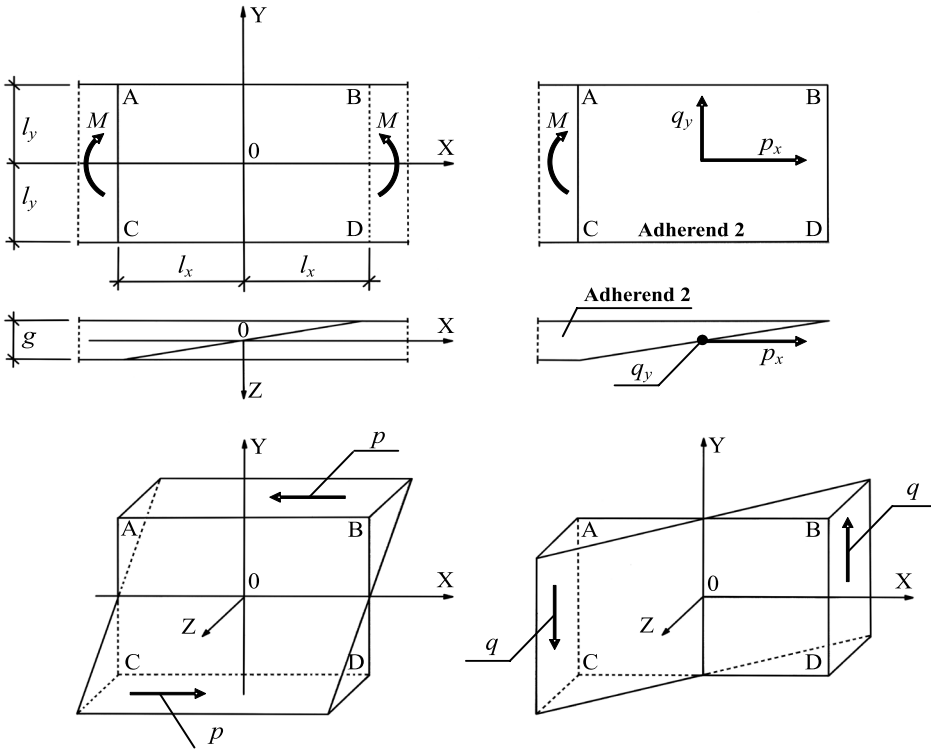


Fig. 7. Approximate distribution of stresses in the adhesive

The normal stress σ_{1x} in adherend 1 (fig. 7a) is approximately constant with respect to x and linear in y . It can be given by:

$$\sigma_{1x}(x, y) = -\frac{3My}{2gl_y^3} \quad (28)$$

Let us also assume that:

$$\sigma_{1y}(x, y) = 0 \quad (29)$$

The detailed analysis of the stress distributions in the adhesive and the adherends of the considered scarf joint loaded by a moment, which was the basis for the above assumed approximations, was carried out in the paper [Rapp 2014].

The displacements u_1 and v_1 in adherend 1 of the scarf joint fulfil the following set of equations:

$$\frac{E_x}{1-\nu_{xy}\nu_{yx}} \frac{\partial u_1}{\partial x} + \frac{\nu_{xy}E_x}{1-\nu_{xy}\nu_{yx}} \frac{\partial v_1}{\partial y} = -\frac{3My}{2gl^3} \quad (30)$$

$$\frac{\nu_{yx}E_y}{1-\nu_{xy}\nu_{yx}} \frac{\partial u_1}{\partial x} + \frac{E_y}{1-\nu_{xy}\nu_{yx}} \frac{\partial v_1}{\partial y} = 0 \quad (31)$$

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = \frac{q(x-l_x)}{gG_{xy}} \quad (32)$$

which is obtained after substitution of relations (27) – (29) with the constitutive equations (7) – (9).

Solving equations (30) and (31) in terms of $\partial u_1/\partial x$ and $\partial v_1/\partial y$, one gets:

$$\frac{\partial u_1}{\partial x} = -\frac{3My}{2E_x gl^3}, \quad \frac{\partial v_1}{\partial y} = \frac{3M\nu_{yx}y}{2E_x gl^3}$$

and after integration the result is:

$$u_1 = -\frac{3Mxy}{2E_x gl^3} + f(y), \quad v_1 = \frac{3M\nu_{yx}y^2}{4E_x gl^3} + h(x) \quad (33)$$

where f is an arbitrary function of the variable y and h is an arbitrary function of the variable x .

Substitution of these relations to expression (32) leads to:

$$-\frac{df(y)}{dy} = \frac{dh(x)}{dx} - \frac{3Mx}{2E_x gl^3} - \frac{q(x-l_x)}{gG_{xy}} \quad (34)$$

The left-hand side of equality (34) represents a function of the variable y , while the right-hand side – a function of the variable x . Thus, both sides of (34) must be constant functions. Hence,

$$-\frac{df(y)}{dy} = A, \quad \frac{dh(x)}{dx} - \frac{3Mx}{2E_x g l_y^3} - \frac{q(x-l_x)}{gG_{xy}} = A \quad (35)$$

where A is an arbitrary constant. The functions f and h can be found by integration of (35):

$$f(y) = -Ay + B, \quad h(x) = \frac{3Mx^2}{4gE_x l_y^3} + \frac{qx^2}{2gG_{xy}} - \frac{ql_x x}{gG_{xy}} + Ax + C$$

where B and C are arbitrary constants. Substitution of these functions to the formulae (33) gives the displacement functions u_1 and v_1 in the form:

$$u_1(x, y) = -\frac{3Mxy}{2gE_x l_y^3} - Ay + B \quad (36)$$

$$v_1(x, y) = \frac{3Mv_{yx}y^2}{4gE_x l_y^3} + \frac{3Mx^2}{4gE_x l_y^3} + \frac{qx^2}{2gG_{xy}} - \frac{ql_x x}{gG_{xy}} + Ax + C \quad (37)$$

where A , B , and C are arbitrary constants.

It is assumed that the scarf joint loaded by moments features a deformation mode symmetric about the Y axis. This yields the conditions:

$$u_2(x, y) = -u_1(-x, y), \quad v_2(x, y) = v_1(-x, y)$$

leading to:

$$u_2(x, y) = -\frac{3Mxy}{2gE_x l_y^3} + Ay + B \quad (38)$$

$$v_2(x, y) = \frac{3Mv_{yx}y^2}{4gE_x l_y^3} + \frac{3Mx^2}{4gE_x l_y^3} + \frac{qx^2}{2gG_{xy}} + \frac{ql_x x}{gG_{xy}} - Ax + C \quad (39)$$

Stresses in the adhesive depend on the displacement difference between adherends 1 and 2. The constitutive relation for the adhesive in the Y direction takes the form:

$$\frac{t \cos \varphi_x}{G_s} q_y(x, y) = v_1(x, y) - v_2(x, y) \quad (40)$$

Equation (40), after substitution of the second relation (24) and the formulae (37) and (39), yields the constant A :

$$A = q \cdot \left(\frac{l_x}{gG_{xy}} + \frac{t \cos \varphi_x}{2G_s l_x} \right) \quad (41)$$

The constitutive relation in the X direction reads:

$$-p_x(x, y) \cos \varphi_x = \frac{G_s}{t} \delta_u (u_1(x, y) - u_2(x, y)) \quad (42)$$

where:

$$\delta_u = \frac{1 + \operatorname{tg}^2 \varphi_x}{1 + \frac{G_s}{E_s} \operatorname{tg}^2 \varphi_x}$$

Equation (42), after substitution of the first relation (24) and the formulae (36) and (39), yields the constant A :

$$A = p \cdot \frac{t \cos \varphi_x}{2l_y G_s \delta_u} \quad (43)$$

Constants B and C are found using the kinematic boundary conditions. Comparing relations (41) and (43) leads to:

$$p \cdot \frac{t \cos \varphi_x}{2l_y G_s \delta_u} = q \cdot \left(\frac{l_x}{gG_{xy}} + \frac{t \cos \varphi_x}{2G_s l_x} \right) \quad (44)$$

Values p and q follow from relations (25) and (44):

$$p = \frac{3M \left(\frac{2\delta_u l_x G_s}{\operatorname{tg} G_{xy} \cos \varphi_x} + \frac{\delta_u}{l_x} \right)}{4 \left(\frac{2\delta_u l_x^2 l_y^2 G_s}{\operatorname{tg} G_{xy} \cos \varphi_x} + l_x^2 + \delta_u l_y^2 \right)} \quad (45)$$

$$q = \frac{3M}{4l_y \left(\frac{2\delta_u l_x^2 l_y^2 G_s}{\operatorname{tg} G_{xy} \cos \varphi_x} + l_x^2 + \delta_u l_y^2 \right)} \quad (46)$$

Having found p and q , from (24) one can determine approximate stress distributions in the adhesive of the joint loaded by moments:

$$\tau_x(x, y) = -\frac{p}{l_y} y \cos^2 \varphi_x \quad (47)$$

$$\sigma_N(x, y) = -\frac{p}{l_y} y \sin \varphi_x \cos \varphi_x \quad (48)$$

$$\tau_y(x, y) = \frac{q}{l_x} x \cos \varphi_x \quad (49)$$

Complete stress sets in the adhesive resulting from the loading by forces N , T , and M are determined using the superposition rule leading to the following relations:

$$\tau_x(x, y) = -\frac{p}{l_y} y \cos^2 \varphi_x + \frac{N}{A} \sin \varphi_x \cos \varphi_x \quad (50)$$

$$\sigma_N(x, y) = -\frac{p}{l_y} y \sin \varphi_x \cos \varphi_x + \frac{N}{A} \sin^2 \varphi_x \quad (51)$$

$$\tau_y(x, y) = \frac{q}{l_x} x \cos \varphi_x + \frac{3T(l_y^2 - y^2)}{4l_y^3 \sqrt{4l_x^2 + g^2}} \quad (52)$$

and complete stress sets in the adherends of the scarf joint are given by:

$$\sigma_{1x}(x, y) = \sigma_{2x}(x, y) = \frac{N}{A} - \frac{3My}{2gl_y^3} \quad (53)$$

$$\sigma_{1y}(x, y) = \sigma_{2y}(x, y) = 0 \quad (54)$$

$$\tau_{1xy}(x, y) = \frac{3T(l_y^2 - y^2)}{4gl_y^3} + \frac{q(x - l_x)}{g} \quad (55)$$

$$\tau_{2xy}(x, y) = \frac{3T(l_y^2 - y^2)}{4gl_y^3} - \frac{q(x - l_x)}{g} \quad (56)$$

where p and q are defined in (45) and (46).

Numerical example

The formulae for stresses in the adhesive and the adherends of the scarf joint loaded by the normal force N and the shear force T coincide with the ones resulting from the theory of elasticity. Thus, it is only necessary to verify the accuracy of the suggested method in the case of the joint loaded by moments M .

Let us consider a scarf joint loaded by the moments $M = 1 \text{ N}\cdot\text{cm}$ according to fig. 4, made of spruce wood with the material parameters $E_x = 1.2 \cdot 10^6 \text{ N/cm}^2$, $E_y = 0.8 \cdot 10^5 \text{ N/cm}^2$, $G_{xy} = 0.6 \cdot 10^5 \text{ N/cm}^2$, $\nu_{xy} = 0.03$, and $\nu_{yx} = 0.45$, the dimensions $l_x = 22.5 \text{ cm}$, $l_y = 10.25 \text{ cm}$, and $g = 4.5 \text{ cm}$. The adhesive thickness is assumed as $t = 0.05 \text{ cm}$ and its material constants $E_s = 1.215 \cdot 10^5 \text{ N/cm}^2$, $G_s = 0.45 \cdot 10^5 \text{ N/cm}^2$, and $\nu_s = 0.35$. These data yield $\text{tg}\varphi_x = 0.1$; $\cos\varphi_x = 1/\sqrt{101}$ and $\delta_u = 1,00627$. Substitution of the data to relations (45) and (46) leads to $p = 3.16824 \cdot 10^{-4}$ and $q = 2.03704 \cdot 10^{-7}$. For $y = l_y$ and $x = l_x$, formulae (47) and (49) make it possible to determine the values of τ_x and τ_y and to compare them with appropriate mean values $\tau_{x,mean}$ and $\tau_{y,mean}$ computed from the solution of the linear theory of elasticity (fig. 5). The results are presented in table 3.

Table 3. Comparison of stresses in the adhesive of a scarf joint loaded by moments

Adhesive edge	Approximate method	Linear theory of elasticity	Error [%]
$y = l_y$	$\tau_x = -3.1369 \cdot 10^{-4} \text{ N/cm}^2$	$\tau_{x,mean} = -3.1185 \cdot 10^{-4} \text{ N/cm}^2$	-0.59
$x = l_x$	$\tau_y = 2.0269 \cdot 10^{-7} \text{ N/cm}^2$	$\tau_{y,mean} = 2.1779 \cdot 10^{-7} \text{ N/cm}^2$	6.93

$\tau_{y,mean} = 2.1779 \cdot 10^{-7} \text{ N/cm}^2$ is the mean value from the entire edge $x = l_x$

Stress τ_x in the adhesive from the approximate method has an accuracy of -0.59% , while stress $\tau_y - 6.93\%$. However, stress τ_y is approx. 10^3 times smaller than τ_x . Thus, in order to correctly assess the accuracy of the approximate method, it is better to analyse the extreme resultant stress at the corner of the adhesive and compare it with the extreme mean stress τ_{mean} coming from the equations of the theory of elasticity. For stresses τ and τ_{mean} , one gets:

$$\tau = \sqrt{\tau_x^2(1 + \text{tg}^2\varphi_x) + \tau_y^2}$$

$$\tau_{mean} = \sqrt{\tau_{x,mean}^2(1 + \text{tg}^2\varphi_x) + \tau_{y,mean}^2}$$

Substitution of the values given in table 3 to these relations yields:

$$\tau = 3.1525 \cdot 10^{-4} \text{ N/cm}^2$$

$$\tau_{mean} = 3.1341 \cdot 10^{-4} \text{ N/cm}^2,$$

Hence, resultant stress τ from the approximate method is computed with an accuracy of -0.59% with respect to stress τ_{mean} resulting from the exact approach.

In the analysed case, the error in stress τ_y does not practically influence the accuracy of the resultant stress. In cases of different ratios of joint dimensions, errors may vary. However, for $l_x \geq l_y$, stress τ_x remains dominant and the error in stress τ_y has a small influence on the error of the resultant stress.

Thus, it can be concluded, that the approximate method can be considered a sufficiently accurate approach to the analysis from the point of view of engineering practice.

Conclusions

The paper presents an approximate formulation of a numerical model for adhesive scarf joints in wooden beams with an assumption of adhesive deformability. This approximate model is based on conclusions coming from the general solution of the plane linear theory of elasticity. It was assumed that wood is orthotropic and the joint can be loaded by axial forces, shear forces and bending moments. In the framework of the approximate numerical model, analytical relations for stresses in the adhesive, as well as displacements and stresses in the adherends of the scarf joint, have been derived.

It was shown that the scarf joint loaded axially in the case of adherends made of an identical material, the stress states in the adhesive and the adherends, do not depend on the thickness and material parameters and are identical to those in a cross-section at angle φ_x of a continuous element under a uniaxial stress state.

The adhesive parameters influence only the difference between the adherend displacements.

Shear force generates shear stress τ_{xy} in the adherends and shear stress τ_y in the adhesive, which, with a good degree of accuracy – approx. 0.25% – feature parabolic distributions defined by the relations for plain beams well known from the strength of materials.

Based on the exact solution of the linear theory of elasticity for a scarf joint loaded by moments, an approximate method has been proposed to compute displacements and stresses in the adherends and stresses in the adhesive of the scarf joint.

A numerical example proved the high degree of accuracy of the approximate model when compared to the general one formulated within the framework of the plane linear theory of elasticity.

The derived relations are simple and very useful in practical engineering computations.

Currently a broad series of natural scale model experiments on beams with scarf joints is being carried out in the Institute of Structural Engineering of Poznan University of Technology. It is planned that the results of these experiments will be used to formulate the damage criterion for a joint and to verify stress distribu-

tions in adhesive and adherends using the finite element method implemented in Abaqus software.

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